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Wigner Quasiprobability Distribution

Abstract

This paper presents a systematic study of the Wigner quasiprobability distribution as a phase-space formulation of quantum mechanics. Starting from the incompatibility between classical probability theory and quantum mechanics, the Wigner function is derived using Weyl quantization and the Weyl transform. Its key properties, including normalization, correct marginals, reality, and invertibility, are discussed in detail. The work further examines Wigner functions for harmonic oscillator states, superposition states, and mixed states, highlighting the role of quantum interference and negativity. Finally, the connection between Wigner functions, quantum tomography, and the Radon transform is explored. We end with derivation of the time evolution equation of the Wigner function.

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1 WHY CLASSICAL PROBABILITY CAN'T DESCRIBE QUANTUM STATES

1.1 Fundamental Incompatibility

Can quantum states be represented as classical probability distributions over phase space?

The answer is emphatically *NO*. Classical mechanics assumes phase space can be resolved arbitrarily finely, but quantum mechanics imposes a minimum phase-space area $\sim \hbar$. This is a fundamental impossibility that forces us beyond classical probability. There is no mapping from quantum states to classical probability distributions over phase space variables that reproduces all quantum predictions.

Theorem 1.1. Impossibility of Joint Probability Distribution
There does not exist a classical probability distribution $f(x,p)$ (satisfying $f \geq 0$, $\int f\,dx\,dp = 1$) that can simultaneously satisfy:

1. **Correct Position Marginal**: $\int f(x,p)\,dp = |\psi(x)|^2$ for all quantum wavefunctions $|\psi\rangle$
2. **Correct Momentum Marginal**: $\int f(x,p)\,dx = |\phi(p)|^2$ where $\phi(p)$ is the Fourier transform of $\psi(x)$
3. **Correct Expectation Values**: For any quantum observable $\hat{A}$, the expectation value computed classically as $\langle A\rangle = \int f(x,p)A(x,p)\,dx\,dp$ matches the quantum expectation $\langle \hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$

1.2 Proof of non-existence of a Positive Joint Phase-Space Distribution

We show that there does not exist a joint probability distribution $f(x,p)$ on phase space satisfying all three conditions above simultaneously:

Step 1: Classical variance constraint

Let $A = x + \lambda p$ for $\lambda \in \mathbb{R}$. Since $f(x,p)$ is a non-negative probability distribution, the variance satisfies:

$$\mathrm{Var}(A) = \langle (A - \langle A\rangle)^2\rangle \geq 0.$$

Expanding,

$$\begin{aligned}\mathrm{Var}(A) &= \mathrm{Var}(x) + \lambda^2\mathrm{Var}(p) + 2\lambda\,\mathrm{Cov}(x,p)\\ &= (\Delta x)^2 + \lambda^2(\Delta p)^2 + 2\lambda\,\mathrm{Cov}(x,p).\end{aligned}$$

Since this quadratic polynomial in λ must be non-negative for all $\lambda \in \mathbb{R}$, its discriminant must satisfy:

$$4\,\mathrm{Cov}(x,p)^2 - 4(\Delta x)^2(\Delta p)^2 \leq 0.$$

Hence,

$$(\Delta x)^2(\Delta p)^2 \geq \mathrm{Cov}(x,p)^2.$$

This is the classical constraint imposed by the existence of a positive joint distribution.

Step 2: Quantum mechanical constraint

Let $\hat{x}, \hat{p}$ be self-adjoint operators on a Hilbert space satisfying

$$[\hat{x}, \hat{p}] = i\hbar.$$

For any normalized state ψ, define expectation values:

$$\langle \hat{x} \rangle = \langle \psi | \hat{x} | \psi \rangle, \quad \langle \hat{p} \rangle = \langle \psi | \hat{p} | \psi \rangle.$$

Define the centered operators:

$$\Delta\hat{x} = \hat{x} - \langle \hat{x} \rangle, \quad \Delta\hat{p} = \hat{p} - \langle \hat{p} \rangle.$$

The variances are:

$$(\Delta x)^2 = \langle (\Delta\hat{x})^2 \rangle, \quad (\Delta p)^2 = \langle (\Delta\hat{p})^2 \rangle.$$

Define the covariance:

$$\mathrm{Cov}(x, p) = \frac{1}{2}\langle \Delta\hat{x}\,\Delta\hat{p} + \Delta\hat{p}\,\Delta\hat{x} \rangle.$$

Using the Cauchy–Schwarz inequality:

$$\|\Delta\hat{x}\psi\|^2 \, \|\Delta\hat{p}\psi\|^2 \geq |\langle \Delta\hat{x}\psi | \Delta\hat{p}\psi \rangle|^2 \,,$$

we obtain:

$$(\Delta x)^2 (\Delta p)^2 \geq |\langle \psi | \Delta\hat{x}\,\Delta\hat{p} | \psi \rangle|^2 \,.$$

Now decompose:

$$\Delta\hat{x}\,\Delta\hat{p} = \frac{1}{2}[\Delta\hat{x}, \Delta\hat{p}] + \frac{1}{2}\{\Delta\hat{x}, \Delta\hat{p}\}.$$

Taking expectation values:

$$\langle \Delta\hat{x}\,\Delta\hat{p} \rangle = \frac{1}{2}\langle [\Delta\hat{x}, \Delta\hat{p}] \rangle + \frac{1}{2}\langle \{\Delta\hat{x}, \Delta\hat{p}\} \rangle.$$

Since

$$[\Delta\hat{x}, \Delta\hat{p}] = [\hat{x}, \hat{p}] = i\hbar,$$

we get:

$$\langle \Delta\hat{x}\,\Delta\hat{p} \rangle = \mathrm{Cov}(x, p) + \frac{i\hbar}{2}.$$

Taking modulus squared:

$$|\langle \Delta\hat{x}\,\Delta\hat{p} \rangle|^2 = \mathrm{Cov}(x, p)^2 + \frac{\hbar^2}{4}.$$

Therefore:

$$(\Delta x)^2 (\Delta p)^2 \geq \mathrm{Cov}(x, p)^2 + \frac{\hbar^2}{4}. \tag{1.2.1}$$

Eq. (1.2.1) is Robertson-Schrödinger Inequality.

Step 3: Contradiction

Comparing:

$$\begin{aligned} \text{Classical:} \quad & (\Delta x)^2(\Delta p)^2 \geq \text{Cov}(x,p)^2 \\ \text{Quantum:} \quad & (\Delta x)^2(\Delta p)^2 \geq \text{Cov}(x,p)^2 + \frac{\hbar^2}{4} \end{aligned}$$

we see that quantum mechanics imposes a strictly stronger lower bound.

Therefore, no positive joint distribution $f(x,p)$ can reproduce all quantum expectation values, since it would violate the quantum uncertainty relation. Any phase-space representation of quantum mechanics must therefore relax at least one of these conditions (e.g. allowing negativity, as in the Wigner function).

1.3 The Resolution: Quasiprobability Distributions

Since classical probability distributions cannot represent quantum states, we must allow distributions with *negative values*.

> **Definition 1.1. Quasiprobability Distribution**
> A function $W(x,p)$ is a *quasiprobability distribution* if it satisfies:
>
> 1. **Can Be Negative**: $W(x,p)$ can be negative or even complex (though for real density operators, W is real)
> 2. **Correct Marginals**:
> $$\int_{-\infty}^{\infty} W(x,p)\,dp = |\psi(x)|^2$$
> $$\int_{-\infty}^{\infty} W(x,p)\,dx = |\phi(p)|^2$$
> 3. **Correct Expectation Values**: For Weyl symbol $A_W(x,p)$ of operator $\hat{A}$:
> $$\langle \hat{A} \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x,p)A_W(x,p)\,dx\,dp$$
> 4. **Unique Invertibility**: The quasiprobability distribution uniquely determines the density operator $\hat{\rho}$
> 5. **Real-Valued for Real $\hat{\rho}$**: If $\hat{\rho}$ is Hermitian (real eigenvalues), then W is real-valued (though not necessarily non-negative)

$W(x,p)$ is known as *Wigner function* named after physicist *Eugene Wigner*. The *Wigner function* is the unique symmetric choice satisfying all these requirements. It treats position and momentum symmetrically (respecting the symplectic structure of phase space) and is therefore the most natural and commonly used quasiprobability distribution.

1.4 Why Negativity Is Essential, Not Pathological

The negative regions of Wigner functions are *not bugs—they're features*:

1. **Encode Quantum Interference**: Negative regions appear precisely where quantum interference (superposition of distinct states) produces oscillations

2. **Measure Quantumness**: The integral of the negative part quantifies "how quantum" a state is:

$$\mathcal{V} = \frac{1}{2}\left(\int |W|\, dx\, dp - 1\right) = \text{(negativity volume)}$$

This is a quantitative measure of non-classicality

3. **Enable Quantum Advantage**: In quantum information theory, Wigner negativity is a *resource for quantum computation.* States with negative Wigner functions can simulate quantum algorithms that positive-only distributions cannot

4. **Necessary for Completeness**: Without allowing negative values, it's mathematically impossible to simultaneously represent correct position and momentum marginals and correct expectation values.

2 DENSITY MATRICES IN POSITION REPRESENTATION

2.1 Definition in Coordinate Basis

In previous sections, we worked with the abstract density operator $\hat{\rho}$. To understand quantum states in position space and visualize phase-space distributions later, we need the *density matrix representation* in coordinate basis.

> **Definition 2.1. Density Matrix in Position Basis**
> The density matrix in the position basis is:
>
> $$\rho(x, x') = \langle x|\hat{\rho}|x'\rangle$$
>
> This is a function of *two position variables* x and x'. It represents the amplitude for the quantum system to be at position x and simultaneously at position x' (in a quantum mechanical sense).

Properties of Density Matrix:

1. **Hermiticity**: $\rho(x', x) = \rho^*(x, x')$

Proof: Since $\hat{\rho}$ is Hermitian:

$$\rho(x', x) = \langle x'|\hat{\rho}|x\rangle = \langle x|\hat{\rho}^\dagger|x'\rangle^* = \langle x|\hat{\rho}|x'\rangle^* = \rho^*(x, x') \quad \square$$

2. **Normalization**:

$$\int_{-\infty}^{\infty} \rho(x, x)\, dx = \text{Tr}(\hat{\rho}) = 1 \tag{2.1.1}$$

3. **Expectation Value**: For any operator $\hat{A}$:

$$\langle \hat{A}\rangle = \text{Tr}(\hat{\rho}\hat{A}) \tag{2.1.2}$$

Note: Refer to the appendix for the proofs of 2nd and 3rd property.

2.2 Interpretation of Diagonal and Off-Diagonal Elements

Diagonal Elements ($x = x'$):

$$\rho(x, x) = \langle x|\hat{\rho}|x\rangle$$

Meaning: $\rho(x, x)$ is a *probability density*. The quantity $\rho(x, x)dx$ represents the probability of finding the particle in the infinitesimal interval $[x, x + dx]$.

The *marginal position probability distribution* is:

$$P(x) = \rho(x, x)$$

This satisfies:

$$\int_{-\infty}^{\infty} P(x)\, dx = \int_{-\infty}^{\infty} \rho(x, x)\, dx = 1$$

For a pure state $|\psi\rangle$, we have $P(x) = |\psi(x)|^2$.

Off-Diagonal Elements ($x \neq x'$):

$$\rho(x, x') \quad \text{with} \quad x \neq x'$$

Meaning: These elements represent *quantum coherence* between different spatial locations. They encode the interference pattern between different spatial components of the wavefunction.

Key properties:

- For a *pure state* $|\psi\rangle$: $\rho(x, x') = \psi(x)\psi^*(x')$ (off-diagonal terms are non-zero)

- For a *completely decohered mixture*: all off-diagonal terms vanish, $\rho(x, x') = 0$ for $x \neq x'$

Physical Significance: Off-diagonal elements are responsible for *quantum interference*. When we compute expectation values, these cross-terms produce the distinctive quantum effects impossible in classical mechanics.

3 THE OPERATOR ORDERING PROBLEM AND QUANTIZATION

3.1 The Fundamental Ambiguity

From classical mechanics to quantum mechanics, we must *"quantize"* classical observables - convert them from functions to operators. This process reveals a fundamental ambiguity not present classically.

In classical mechanics, position and momentum are *commuting numbers*:

$$xp = px \quad \text{(classically)}$$

In quantum mechanics, they are *non-commuting operators*:

$$[\hat{x}, \hat{p}] = i\hbar \neq 0 \quad \text{(quantum)}$$

This non-commutativity creates an *ordering problem* when quantizing classical functions.

3.2 Example: Quantizing the Classical Observable xp

Consider the classical observable $A(x,p) = xp$. When we quantize, which operator should represent this?

Option 1 - Left Ordering:

$$\hat{A}_L = \hat{x}\hat{p}$$

This means: first apply momentum operator, then position.

Option 2 - Right Ordering:

$$\hat{A}_R = \hat{p}\hat{x}$$

This means: first apply position operator, then momentum.

Option 3 - Symmetric (Weyl) Ordering:

$$\hat{A}_S = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$$

Average of left and right orderings.

3.3 How the Orderings Differ

From the canonical commutation relation:

$$\hat{x}\hat{p} = \hat{p}\hat{x} + [\hat{x},\hat{p}] = \hat{p}\hat{x} + i\hbar$$

Therefore:

$$\hat{A}_L = \hat{x}\hat{p} = \hat{p}\hat{x} + i\hbar$$

$$\hat{A}_S = \frac{1}{2}(\hat{p}\hat{x} + i\hbar + \hat{p}\hat{x}) = \hat{p}\hat{x} + \frac{i\hbar}{2}$$

$$\hat{A}_R = \hat{p}\hat{x}$$

The three orderings differ by terms proportional to $\hbar$:

$$\hat{A}_L - \hat{A}_S = \frac{i\hbar}{2}, \quad \hat{A}_S - \hat{A}_R = \frac{i\hbar}{2}$$

For macroscopic systems ($\hbar \to 0$), all three converge to the same result. But for quantum systems, the choice matters!

3.4 Which Ordering Scheme Should We Use?

Requirements for a Good Quantization Scheme:

1. *Hermiticity*: If the classical observable $A(x,p)$ is real-valued, the quantum operator $\hat{A}$ should be self-adjoint: $\hat{A}^\dagger = \hat{A}$

2. *Uniqueness*: There should be one preferred quantization, not multiple ambiguous choices

3. *Covariance*: The scheme should respect symmetries - canonical transformations in phase space should correspond to unitary transformations in Hilbert space

4. *Symmetry in x and p*: Position and momentum should be treated equally, without preferential role

The Weyl Quantization Scheme satisfies all four requirements and is the unique scheme doing so.

3.5 Definition of Weyl Quantization

Definition 3.1. Weyl Quantization Scheme
A classical observable $A(x,p)$ is quantized to an operator via the *Weyl correspondence*:

$$\hat{A} = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dp\, A(x,p)\, \hat{W}(x,p)$$

where the *Weyl displacement operator* is:

$$\hat{W}(x,p) = \exp\left(\frac{i}{\hbar}(p\hat{x} - x\hat{p})\right)$$

Note the Key Feature: The exponent $(p\hat{x} - x\hat{p})$ is *antisymmetric* in (x,p), preserving the symplectic structure of phase space.

3.6 Important Properties of Weyl Quantization

Property 1 - Positional Operator:

For the classical observable $A(x,p) = x$,

$$\hat{A} = \frac{1}{2\pi\hbar}\int dx\, dp\, x\, e^{\frac{i}{\hbar}(p\hat{x} - x\hat{p})}$$

Using the Baker–Campbell–Hausdorff decomposition,

$$e^{\frac{i}{\hbar}(p\hat{x} - x\hat{p})} = e^{-\frac{i}{2\hbar}px} e^{\frac{i}{\hbar}p\hat{x}} e^{-\frac{i}{\hbar}x\hat{p}}$$

we evaluate the matrix element in the position basis:

$$\langle q'|\hat{W}(x,p)|q\rangle = e^{-\frac{i}{2\hbar}px}\langle q'|e^{\frac{i}{\hbar}p\hat{x}} e^{-\frac{i}{\hbar}x\hat{p}}|q\rangle$$

Using

$$e^{-\frac{i}{\hbar}x\hat{p}}|q\rangle = |q + x\rangle$$

and

$$e^{\frac{i}{\hbar}p\hat{x}}|q\rangle = e^{\frac{i}{\hbar}pq}|q\rangle$$

gives

$$\langle q'|\hat{W}(x,p)|q\rangle = e^{\frac{i}{\hbar}p(q - \frac{x}{2})}\delta(q' - q - x)$$

Therefore,

$$\langle q'|\hat{A}|q\rangle = \frac{1}{2\pi\hbar}\int dx\, dp\, x\, e^{\frac{i}{\hbar}p(q - \frac{x}{2})}\delta(q' - q - x)$$

Using the Fourier identity

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\, e^{\frac{i}{\hbar}pa} = \delta(a)$$

we obtain

$$\langle q'|\hat{A}|q\rangle = \int dx\, x\, \delta\left(q - \frac{x}{2}\right) \delta(q' - q - x)$$

Evaluating the delta functions gives

$$\langle q'|\hat{A}|q\rangle = q\, \delta(q' - q)$$

Since

$$\langle q'|\hat{x}|q\rangle = q\, \delta(q' - q)$$

we conclude that

$$\hat{A} = \hat{x}$$

This reproduces the position operator correctly.

Property 2 - Momentum Operator:

Now consider the classical observable $A(x, p) = p$:

$$\hat{A} = \frac{1}{2\pi\hbar} \int dx\, dp\, p\, e^{\frac{i}{\hbar}(p\hat{x} - x\hat{p})}$$

Proceeding similarly,

$$\langle q'|\hat{A}|q\rangle = \frac{1}{2\pi\hbar} \int dx\, dp\, p\, e^{\frac{i}{\hbar}p(q - \frac{x}{2})} \delta(q' - q - x)$$

Using

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp\, p\, e^{\frac{i}{\hbar}pa} = -i\hbar\, \delta'(a)$$

gives

$$\langle q'|\hat{A}|q\rangle = -i\hbar \int dx\, \delta'\left(q - \frac{x}{2}\right) \delta(q' - q - x)$$

Evaluating the derivative of the delta distribution,

$$\langle q'|\hat{A}|q\rangle = -i\hbar \frac{\partial}{\partial q} \delta(q' - q)$$

But this is precisely the coordinate-space kernel of the momentum operator:

$$\langle q'|\hat{p}|q\rangle = -i\hbar \frac{\partial}{\partial q} \delta(q' - q)$$

Hence,

$$\hat{A} = \hat{p}$$

4 THE WEYL TRANSFORM

4.1 Coordinate Transformation

We want to go from (q, q') coordinates (two position arguments) to (x, ξ) coordinates (midpoint and relative).

Definition 4.1. Coordinate Change

$$x = \frac{q+q'}{2} \quad \text{(midpoint)}$$

$$\xi = q - q' \quad \text{(relative coordinate)}$$

Inverse Relations:

$$q = x + \frac{\xi}{2}$$

$$q' = x - \frac{\xi}{2}$$

4.2 Jacobian of the Transformation

The Jacobian determinant of this transformation is exactly 1:

$$J = \begin{vmatrix} \frac{\partial q}{\partial x} & \frac{\partial q}{\partial \xi} \\ \frac{\partial q'}{\partial x} & \frac{\partial q'}{\partial \xi} \end{vmatrix} = \begin{vmatrix} 1 & 1/2 \\ 1 & -1/2 \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1$$

Therefore: $|J| = 1$

This means: $dq\, dq' = dx\, d\xi$ (area element is preserved)

4.3 Why This Coordinates Are Natural for Quantum Mechanics

Interpretation:

- **Midpoint** x: The position where we evaluate the distribution

- **Relative** ξ: The "separation"

- how far apart the two matrix element arguments are

This naturally encodes *quantum interference*: only nearby positions (small ξ) have significant off-diagonal matrix elements. The Fourier transform of these oscillations (in ξ) produces the momentum dependence.

4.4 Fourier Transform in Relative Coordinate

When we Fourier-transform in the relative coordinate ξ:

$$\int d\xi\, e^{-ip\xi/\hbar}\, \rho(x + \xi/2, x - \xi/2)$$

We're transforming oscillations encoded in the separation ξ into momentum information. The *frequency of oscillation* (as ξ varies) directly determines the momentum p.

4.5 Weyl Transform Definition

Definition 4.2. Weyl Transform - Operator to Function
For an operator $\hat{A}$, its *Weyl symbol* is:

$$A_W(x,p) = \int_{-\infty}^{\infty} d\xi\, e^{-ip\xi/\hbar} \left\langle x + \frac{\xi}{2} \middle| \hat{A} \middle| x - \frac{\xi}{2} \right\rangle$$

This extracts the phase-space function from an operator.

4.6 Key Features of the Weyl Symbol

Feature 1 - Midpoint and relative Coordinates:

Note the use of:

- **Midpoint**: $x =$ (position where we evaluate A_W)

- **Relative coordinate**: $\xi =$ (separation between matrix element arguments)

This *symmetric choice* is essential for Weyl's uniqueness.

Feature 2 - For Position Observable:

$$\hat{A} = \hat{x}$$

$$\begin{aligned} x_W &= \int_{-\infty}^{\infty} d\xi\, e^{-ip\xi/\hbar} \left\langle x + \frac{\xi}{2} |\hat{x}| x - \frac{\xi}{2} \right\rangle \\ &= \int d\xi\, e^{-ip\xi/\hbar} \cdot \left(x - \frac{\xi}{2} \right) \cdot \delta\left(\xi\right) \\ &= x \end{aligned}$$

The Weyl symbol of position is just the position variable.

Feature 3 - For Momentum Observable:

$$\hat{A} = \hat{p} = -i\hbar \frac{d}{dx}$$

$$\begin{aligned} p_W(x,p) &= \int_{-\infty}^{\infty} d\xi\; e^{-ip\xi/\hbar} \left\langle x + \frac{\xi}{2} \middle| \hat{p} \middle| x - \frac{\xi}{2} \right\rangle \\ &= \int d\xi\; e^{-ip\xi/\hbar} \left(-i\hbar \frac{d}{dx} \right) \left\langle x + \frac{\xi}{2} \middle| x - \frac{\xi}{2} \right\rangle \\ &= \int d\xi\; e^{-ip\xi/\hbar} \left(-i\hbar \frac{d}{dx} \right) \delta(\xi) \end{aligned}$$

Using

$$\frac{d}{dx}\delta(\xi) = -\frac{d}{d\xi}\delta(\xi),$$

we get

$$p_W(x,p) = \int d\xi\; e^{-ip\xi/\hbar} (i\hbar) \frac{d}{d\xi} \delta(\xi)$$

Integrating by parts:

$$\begin{aligned} p_W(x,p) &= \int d\xi\ \delta(\xi)\, p\, e^{-ip\xi/\hbar} \\ &= p \end{aligned}$$

The Weyl symbol of momentum is just the momentum variable.

Feature 3 - Inverse Weyl Transform:

Starting from the Weyl transform,

$$A_W(x,p) = \int_{-\infty}^{\infty} d\xi\ e^{-ip\xi/\hbar} \left\langle x + \frac{\xi}{2} \middle| \hat{A} \middle| x - \frac{\xi}{2} \right\rangle$$

define

$$q = x + \frac{\xi}{2}, \qquad q' = x - \frac{\xi}{2}$$

so that

$$x = \frac{q+q'}{2}, \qquad \xi = q - q'$$

Then,

$$A_W(x,p) = \int d\xi\ e^{-ip\xi/\hbar} \langle q|\hat{A}|q'\rangle$$

Recognizing this as a Fourier transform in ξ, we use the inverse Fourier transform:

$$f(\xi) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp\ e^{ip\xi/\hbar} F(p)$$

Thus,

$$\begin{aligned} \langle q|\hat{A}|q'\rangle &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp\ e^{\frac{i}{\hbar}p(q-q')} A_W\left(\frac{q+q'}{2}, p\right) \\ \Rightarrow \qquad \langle q'|\hat{A}|q\rangle &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp\ e^{-\frac{i}{\hbar}p(q-q')} A_W\left(\frac{q+q'}{2}, p\right) \end{aligned} \tag{4.6.1}$$

5 EXPECTATION VALUES FROM WIGNER FUNCTIONS

5.1 The Fundamental Formula

We demand Expectation Value Formula to have the following form:

Definition 5.1. Expectation Value Formula

For any observable $\hat{A}$ with Weyl symbol $A_W(x,p)$:

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x,p)\, A_W(x,p)\, dx\, dp \tag{5.1.1}$$

This formula resembles a *classical expectation value*, but with the Wigner function and Weyl symbols replacing classical probability and classical function.

This is the *profound insight*: quantum mechanics in phase space looks like classical mechanics with generalized probabilities and symbols.

5.2 Why Weyl Symbols Matter

The Weyl symbol is *not* arbitrary. It's the unique choice ensuring:

1. Expectation values match quantum mechanics
2. The formula resembles classical probability

Other orderings (left, right) would give different Weyl symbols and more complicated expectation value formulas.

6 WIGNER FUNCTION DERIVATION

We now combine all previous material into the *derivation of the Wigner function* from first principles.

Step 1: Start with Expectation Value Formula

Using Eq. (2.1.2), for any observable $\hat{A}$, the expectation value is:

$$\langle \hat{A} \rangle = \mathrm{Tr}(\hat{\rho}\hat{A}) = \int_{-\infty}^{\infty} dq\, \langle q|\hat{\rho}\hat{A}|q\rangle$$

where $W(x,p)$ is the *Wigner* phase-space function and A_W is the Weyl symbol.

Step 2: Using resolution of identity

Using resolution of identity:

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} dq\, dq'\, \langle q|\hat{\rho}|q'\rangle\langle q'|\hat{A}|q\rangle$$

Step 3: Switching to Midpoint and Relative coordinates (See Definition 4.1)

Change variables: $(q,q') \to \left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right)$ with $dq\,dq' = dx\,d\xi$:

$$\langle \hat{A} \rangle = \int dx\, d\xi\, \rho(x+\xi/2, x-\xi/2)\langle x-\xi/2|\hat{A}|x+\xi/2\rangle \tag{6.1}$$

Step 4: Inserting Inverse Weyl Symbol at Midpoint

Substituting Eq. (4.6.1) in Eq. (6.1), we have:

$$\langle \hat{A} \rangle = \int dx\, d\xi\, \rho(x+\xi/2, x-\xi/2) \int \frac{dp}{2\pi\hbar} e^{-ip\xi/\hbar} A_W(x,p)$$

Step 5: Exchanging order of Integration

$$\langle \hat{A} \rangle = \int dx \, \frac{dp}{2\pi\hbar} A_W(x,p) \int d\xi \, e^{-ip\xi/\hbar} \rho(x+\xi/2, x-\xi/2) \tag{6.2}$$

We demand Eq. (6.2) to have the form of Eq. (5.1.1). Thus we define the Winger function as follows:

Definition 6.1. Wigner Function

$$W(x,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \, e^{-ip\xi/\hbar} \rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right) \tag{6.3}$$

Or equivalently:

$$W(x,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \, e^{-ip\xi/\hbar} \left\langle x+\frac{\xi}{2} \, |\hat{\rho}| \, x-\frac{\xi}{2} \right\rangle$$

We've shown that the Wigner function:

1. *Emerges naturally* from requiring expectation values to have a specific form
2. *Is uniquely determined* by the density operator
3. *Uses symmetrical coordinates* (midpoint x, relative ξ) for Weyl covariance

7 NORMALIZATION OF THE WIGNER FUNCTION

7.1 Statement of Normalization

Theorem 7.1. Normalization of Wigner Function

The Wigner function satisfies:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x,p) \, dx \, dp = 1$$

This ensures that the Wigner function integrates to unity, consistent with probability conservation.

7.2 Proof of Normalization

Proof:
Step 1: Start with the Wigner function definition:

$$W(x,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \, e^{-ip\xi/\hbar} \rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right)$$

Step 2: Integrating over both x and p:

$$\int dx \, dp \, W(x,p) = \int dx \, dp \, \frac{1}{2\pi\hbar} \int d\xi \, e^{-ip\xi/\hbar} \rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right)$$

Step 3: Exchanging order of integration:

$$\int dx\, dp\, W(x,p) = \int d\xi\, \rho\left(x + \frac{\xi}{2}, x - \frac{\xi}{2}\right) \int dx\, \frac{1}{2\pi\hbar} \int dp\, e^{-ip\xi/\hbar} \tag{7.2.1}$$

Step 4: Evaluating the momentum integral using the delta function:

$$\int_{-\infty}^{\infty} dp\, e^{-ip\xi/\hbar} = 2\pi\hbar\delta(\xi) \tag{7.2.2}$$

Step 5: Substituting Eq. (7.2.2) in Eq. (7.2.1), we get:

$$\begin{aligned}\int dx\, dp\, W(x,p) &= \int d\xi\, \rho\left(x + \frac{\xi}{2}, x - \frac{\xi}{2}\right) \int dx\, \frac{1}{2\pi\hbar} \cdot 2\pi\hbar\delta(\xi) \\ &= \int d\xi\, \delta(\xi) \int dx\, \rho\left(x + \frac{\xi}{2}, x - \frac{\xi}{2}\right) \\ &= \int dx\, \rho(x,x) \\ &= \mathrm{Tr}(\hat{\rho}) = 1 && \text{[Using Eq. (2.1.1)]} \quad \square\end{aligned}$$

since the trace of the density operator is the normalization condition for quantum states.

7.3 Physical Interpretation

The normalization $\int W\, dx\, dp = 1$ ensures that the Wigner function represents a *normalized probability distribution* in an extended sense. While W may be negative (unlike classical probabilities), it does conserve total "weight" integrated over all phase space.

8 CORRECT MARGINAL DISTRIBUTIONS

8.1 Statement of Marginal Conditions

Theorem 8.1. Marginal Distributions of Wigner Function

For a pure state, the Wigner function has marginals given by:

$$\int_{-\infty}^{\infty} W(x,p)\, dp = |\psi(x)|^2 \quad \text{(position marginal)}$$

$$\int_{-\infty}^{\infty} W(x,p)\, dx = |\phi(p)|^2 \quad \text{(momentum marginal)}$$

where $\psi(x) = \langle x|\psi\rangle$ is the position-space wavefunction and $\tilde{\psi}(p) = \phi(p) = \langle p|\psi\rangle$ is the momentum-space wavefunction (fourier transform of $\psi(x)$).

8.2 Proof of Position Marginal

Proof:

Step 1: Integrating Wigner function over momentum:

$$\int_{-\infty}^{\infty} W(x,p)\,dp = \int dp\,\frac{1}{2\pi\hbar}\int d\xi\, e^{-ip\xi/\hbar}\rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right)$$

Step 2: Exchanging order of integration:

$$\int_{-\infty}^{\infty} W(x,p)\,dp = \frac{1}{2\pi\hbar}\int d\xi\,\rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right)\int dp\, e^{-ip\xi/\hbar} \tag{8.2.1}$$

Step 3: Evaluating momentum integral:

$$\int dp\, e^{-ip\xi/\hbar} = 2\pi\hbar\delta(\xi) \tag{8.2.2}$$

Step 4: Substituting Eq. (8.2.1) in Eq. (8.2.1), we get:

$$\begin{aligned}\int_{-\infty}^{\infty} W(x,p)\,dp &= \int d\xi\,\rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right)\delta(\xi)\\ &= \rho(x,x)\\ &= \langle x|\hat{\rho}|x\rangle\\ &= \langle x|\psi\rangle\langle\psi|x\rangle = |\psi(x)|^2 \qquad \text{[for pure state: } \hat{\rho} = |\psi\rangle\langle\psi|] \qquad \square\end{aligned}$$

8.3 Proof of Momentum Marginal

Proof:

Step 1: Integrating Wigner function over position:

$$\int_{-\infty}^{\infty} W(x,p)\,dx = \int dx\,\frac{1}{2\pi\hbar}\int d\xi\, e^{-ip\xi/\hbar}\,\rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right)$$

Step 2: Exchanging order of integration:

$$\int_{-\infty}^{\infty} W(x,p)\,dx = \frac{1}{2\pi\hbar}\int d\xi\, e^{-ip\xi/\hbar}\int dx\,\rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right) \tag{8.3.1}$$

Step 4: Recognizing momentum-space density matrix:
Using the Fourier transform relation:

$$\int dx\,\rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right) = \int dp'\, e^{ip'\xi/\hbar}\,\rho(p',p') \tag{8.3.2}$$

Substituting Eq. (8.3.2) into Eq. (8.3.1):

$$\int W(x,p)\,dx = \frac{1}{2\pi\hbar}\int d\xi\, e^{-ip\xi/\hbar}\int dp'\, e^{ip'\xi/\hbar}\,\rho(p',p') \tag{8.3.3}$$

Step 5: Evaluating the ξ-integral:

$$\int d\xi\, e^{-i(p-p')\xi/\hbar} = 2\pi\hbar\,\delta(p-p') \tag{8.3.4}$$

Step 6: Substituting Eq. (8.3.4) into Eq. (8.3.3), we get:

$$\begin{aligned}\int_{-\infty}^{\infty} W(x,p)\,dx &= \int dp'\,\rho(p',p')\,\delta(p-p') \\ &= \rho(p,p) \\ &= \langle p|\hat{\rho}|p\rangle \\ = &= |\phi(p)|^2 \quad \text{[for pure state: } \hat{\rho} = |\psi\rangle\langle\psi|\text{]} \\ &= |\tilde{\psi}(p)|^2 \qquad \square\end{aligned}$$

8.4 Importance of Correct Marginals

The fact that the Wigner function *automatically gives correct marginals* is *extraordinary*. This was one of our requirements when constructing the quasiprobability distribution. The marginals must match quantum mechanical predictions for position and momentum separately.

This makes the Wigner function *more than just a phase-space function*- it's a window into both position and momentum distributions simultaneously.

9 REALITY AND HERMITICITY PROPERTIES

Theorem 9.1. Reality of Wigner Function

For any Hermitian density operator $\hat{\rho}$ (which all physical quantum states must satisfy):

$$W^*(x,p) = W(x,p) \tag{9.1}$$

The Wigner function is *real-valued*.

Proof:

Step 1: Starting with the Wigner function definition:

$$W(x,p) = \frac{1}{2\pi\hbar}\int d\xi\, e^{-ip\xi/\hbar}\rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right)$$

Step 2: Taking the complex conjugate:

$$W^*(x,p) = \frac{1}{2\pi\hbar}\int d\xi\, e^{ip\xi/\hbar}\rho^*\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right) \tag{9.2}$$

Step 3: Using Hermiticity of $\hat{\rho}$, which gives:

$$\rho^*\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right) = \rho\left(x-\frac{\xi}{2}, x+\frac{\xi}{2}\right) \tag{9.3}$$

Step 4: Substituting Eq. (9.3) in Eq. (9.2), we get:

$$W^*(x,p) = \frac{1}{2\pi\hbar}\int d\xi\, e^{ip\xi/\hbar}\rho\left(x-\frac{\xi}{2}, x+\frac{\xi}{2}\right)$$

Step 5: Changing integration variable $\xi \to -\xi$, we get:

$$\begin{aligned} W^*(x,p) &= \frac{1}{2\pi\hbar}\int d(-\xi)\, e^{-ip(-\xi)/\hbar}\rho\left(x+\frac{-\xi}{2}, x-\frac{-\xi}{2}\right) \\ &= \frac{1}{2\pi\hbar}\int d\xi'\, e^{-ip\xi'/\hbar}\rho\left(x+\frac{\xi'}{2}, x-\frac{\xi'}{2}\right) && \text{(where } \xi' = -\xi\text{)} \\ &= W(x,p) \qquad \square \end{aligned}$$

Therefore, the Wigner function is real-valued.

9.1 Why Reality is Crucial

Reality of the Wigner function is essential for its interpretation:

- Real-valued functions can be visualized easily
- Negative regions have genuine meaning
- Probabilities (though some "negative") are real quantities

This contrasts with some other quasiprobability distributions (like the P-distribution) which can be complex-valued for nonclassical states. However such complex valued distributions are beyond the scope of this paper.

10 INVERTIBILITY OF THE WIGNER FUNCTION

10.1 Invertibility of Wigner Function

Theorem 10.1. Invertibility of Wigner Function

The Wigner function uniquely determines the density operator. We can recover $\hat{\rho}$ via the *inverse Weyl transform*:

$$\rho(q,q') = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{ip(q-q')/\hbar}\, W\left(\frac{q+q'}{2}, p\right)$$

This shows that the information in W completely specifies ρ.

10.2 Proof of Invertibility

Proof:

Step 1: Starting with the Wigner function definition:

$$W(x,p) = \frac{1}{2\pi\hbar}\int d\xi\, e^{-ip\xi/\hbar}\rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right) \tag{10.2.1}$$

Step 2: This is a Fourier transform in the variable ξ. Applying Fourier inversion formula:

$$\rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right) = \int \frac{dp}{2\pi\hbar}\, e^{ip\xi/\hbar}\, W\left(x,p\right) \tag{10.2.2}$$

Step 3: Substituting Eq. (10.2.2) into Eq. (10.2.1) and setting $q' = x - \xi/2$, so $x = (q+q')/2$ and $\xi = q - q'$:

$$\rho(q,q') = \int \frac{dp}{2\pi\hbar}\, e^{ip(q-q')/\hbar}\, W\left(\frac{q+q'}{2}, p\right) \quad \square$$

10.3 Physical Significance

Invertibility means:

1. **One-to-One Correspondence**: Each density operator has unique Wigner function, and vice versa

2. **Complete Information**: The Wigner function is a complete representation of the quantum state

3. **Reconstruction**: From phase-space measurements (via Wigner function), we can exactly reconstruct the quantum state

This is why quantum state tomography (measuring the Wigner function) can completely determine the state.

11 CLASSIFICATION OF WIGNER FUNCTION FOR DIFFERENT STATES

Let $|n\rangle$ be a complete orthonormal basis. Then the density operator can be expanded as

$$\hat{\rho} = \sum_{m,n} \rho_{mn}|m\rangle\langle n|, \quad \rho_{mn} = \langle m|\hat{\rho}|n\rangle. \tag{11.1}$$

We have,

$$\left\langle x+\frac{\xi}{2}\middle|\hat{\rho}\middle|x-\frac{\xi}{2}\right\rangle = \sum_{m,n}\rho_{mn}\left\langle x+\frac{\xi}{2}\middle|m\right\rangle\left\langle n\middle|x-\frac{\xi}{2}\right\rangle = \sum_{m,n}\rho_{mn}\psi_m\left(x+\frac{\xi}{2}\right)\psi_n^*\left(x-\frac{\xi}{2}\right). \tag{11.2}$$

Using Eq. (11.2) in definition of Winger function, we get:

$$W_{mn}(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\xi e^{-ip\xi/\hbar}\sum_{m,n}\rho_{mn}\psi_m\left(x+\frac{\xi}{2}\right)\psi_n^*\left(x-\frac{\xi}{2}\right). \tag{11.3}$$

Eq. (11.3) is the fundamental building block using which we will classify Wigner function for different states.

11.1 Pure States

If the system is in a pure state $|\psi\rangle$, then

$$\hat{\rho} = |\psi\rangle\langle\psi|.$$

Expanding $|\psi\rangle$ into orthonormal basis $|n\rangle$, we get:

$$|\psi\rangle = \sum_n c_n |n\rangle.$$

Then using Eq. (11.1), we have:

$$\rho_{mn} = c_m c_n^*,$$

and hence using Eq. (11.3), we have:

$$\begin{aligned} W(x,p) &= \sum_{m,n} c_m c_n^* W_{mn}(x,p) \\ &= \sum_n |c_n|^2 W_{nn}(x,p) + \sum_{m \neq n} c_m c_n^* W_{mn}(x,p) \end{aligned} \tag{11.1.1}$$

Eq. (11.1.1) is the Wigner function for *superposition of states.* The first term corresponds to classical probabilities, while the second term represents quantum interference.

It follows from Eq. (11.1.1) that for *Fock State* $|n\rangle$, $\rho_{nn} = |c_n|^2$ and $\rho_{mn} = 0$ for $m \neq n$. The Wigner function of *Fock State* $|n\rangle$ is thus:

$$W(x,p) = \sum_n |c_n|^2 W_{nn}(x,p)$$

11.2 Mixed States

For a statistical mixture,

$$\hat{\rho} = \sum_n w_n |n\rangle\langle n|, \quad w_n \geq 0, \quad \sum_n w_n = 1.$$

Then using Eq. (11.1), we have:

$$\rho_{mn} = w_n \delta_{mn},$$

and hence using Eq. (11.1), we have:

$$W(x,p) = \sum_n w_n W_{nn}(x,p).$$

Observation: Mixed states contain only diagonal contributions and therefore exhibit no interference terms.

11.3 Conclusion

The Wigner function admits the general decomposition

$$W(x,p) = \sum_{m,n} \rho_{mn} W_{mn}(x,p)$$

where ρ_{mn} and $W(x,p)$ are given by Eqs. (11.1) and (11.3) respectively. Thus:

- Diagonal elements ρ_{nn} correspond to classical probability distributions.
- Off-diagonal elements ρ_{mn} $(m \neq n)$ encode quantum coherence and give rise to interference in phase space.

12 DERIVATION OF $W_{mn}(x,p)$ FOR THE HARMONIC OSCILLATOR

We start from the definition of the Wigner function:

$$W_{mn}(x,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \; e^{-ip\xi/\hbar} \psi_m^* \left(x + \frac{\xi}{2}\right) \psi_n \left(x - \frac{\xi}{2}\right) \tag{12.1}$$

Step 1: Harmonic oscillator eigenfunctions

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \, x_0 \sqrt{\pi}}} H_n\left(\frac{x}{x_0}\right) e^{-x^2/(2x_0^2)}, \quad x_0 = \sqrt{\frac{\hbar}{m\omega}} \tag{12.2}$$

Substituting Eq. (12.2) into Eq. (12.1), we get:

$$\begin{aligned} W_{mn}(x,p) = & \frac{1}{2\pi\hbar} \frac{1}{\sqrt{2^{m+n} m! n! \pi x_0^2}} \\ & \times \int_{-\infty}^{\infty} d\xi \; e^{-ip\xi/\hbar} H_m\left(\frac{x+\xi/2}{x_0}\right) H_n\left(\frac{x-\xi/2}{x_0}\right) \\ & \times \exp\left[-\frac{(x+\xi/2)^2 + (x-\xi/2)^2}{2x_0^2}\right] \end{aligned} \tag{12.3}$$

Step 2: Simplifying the exponential

$$(x+\xi/2)^2 + (x-\xi/2)^2 = 2x^2 + \frac{\xi^2}{2}$$

Thus,

$$\exp\left[-\frac{(x+\xi/2)^2 + (x-\xi/2)^2}{2x_0^2}\right] = \exp\left[-\frac{x^2}{x_0^2} - \frac{\xi^2}{4x_0^2}\right] \tag{12.4}$$

Substituting Eq. (12.4) into Eq. (12.3), we get:

$$\begin{aligned} W_{mn}(x,p) = & C \, e^{-x^2/x_0^2} \int_{-\infty}^{\infty} d\xi \; e^{-\xi^2/(4x_0^2)} e^{-ip\xi/\hbar} \\ & \times H_m\left(\frac{x+\xi/2}{x_0}\right) H_n\left(\frac{x-\xi/2}{x_0}\right) \end{aligned} \tag{12.5}$$

Step 3: Changing variables
We define:

$$u = \frac{\xi}{2x_0}, \quad d\xi = 2x_0\, du \tag{12.6}$$

We also define dimensionless variables:

$$\xi' = \frac{x}{x_0}, \quad \pi = \frac{px_0}{\hbar} \tag{12.7}$$

Substituting Eqs. (12.6) and (12.7) in Eq. (12.5), we get:

$$W_{mn} = C' e^{-\xi'^2} \int_{-\infty}^{\infty} du\, e^{-u^2} e^{-2i\pi u}\, H_m(\xi' + u) H_n(\xi' - u) \tag{12.8}$$

Step 4: Using Rodrigues formula

The Rodrigues formula is given by:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Apply Rodriguez formula to both Hermite polynomials,we get:

$$H_m(\xi' + u) = (-1)^m e^{(\xi'+u)^2} \frac{d^m}{d(\xi' + u)^m} e^{-(\xi'+u)^2} \tag{12.9}$$

$$H_n(\xi' - u) = (-1)^n e^{(\xi'-u)^2} \frac{d^n}{d(\xi' - u)^n} e^{-(\xi'-u)^2} \tag{12.10}$$

Inserting Eqs. (12.9) and (12.10) into the integral of Eq. (12.8), the Gaussian factors combine with e^{-u^2}, i.e.,:

$$e^{-u^2} e^{(\xi'+u)^2} e^{(\xi'-u)^2} = e^{2\xi'^2}$$

Thus Eq. (12.8) equates to::

$$\begin{aligned} W_{mn} = C'' e^{-\xi'^2} e^{2\xi'^2} \int du\, e^{-2i\pi u} \\ \times \frac{d^m}{d(\xi' + u)^m} \frac{d^n}{d(\xi' - u)^n} e^{-(\xi'+u)^2 - (\xi'-u)^2} \end{aligned} \tag{12.11}$$

Step 5: Combining exponent again

$$(\xi' + u)^2 + (\xi' - u)^2 = 2\xi'^2 + 2u^2$$

So we have:

$$e^{-(\xi'+u)^2 - (\xi'-u)^2} = e^{-2\xi'^2} e^{-2u^2} \tag{12.12}$$

Substituting Eq. (12.12) into Eq. (12.11) and canceling factors, we get:

$$\begin{aligned} W_{mn} = C''' \int_{-\infty}^{\infty} du\, e^{-2u^2} e^{-2i\pi u} \\ \times \frac{d^m}{d(\xi' + u)^m} \frac{d^n}{d(\xi' - u)^n} 1 \end{aligned} \tag{12.13}$$

Step 6: Moving derivatives using integration by parts
We transfer derivatives onto the Gaussian using integration by parts:

$$\frac{d}{du}e^{-2u^2-2i\pi u} = (-4u-2i\pi)\,e^{-2u^2-2i\pi u}$$

After applying $m+n$ derivatives:

$$\frac{d^{m+n}}{du^{m+n}}e^{-2u^2-2i\pi u} = P_{mn}(u,\pi)\,e^{-2u^2-2i\pi u} \tag{12.14}$$

where P_{mn} is a polynomial.
Substituting Eq. (12.14) into Eq. (12.13), we get:

$$W_{mn} \propto \int_{-\infty}^{\infty} du\; e^{-2u^2-2i\pi u}\; P_{mn}(u,\pi,\xi') \tag{12.15}$$

Step 7: Completing the square

$$-2u^2-2i\pi u = -2\left(u+\frac{i\pi}{2}\right)^2 - \frac{\pi^2}{2}$$

Thus we have,

$$e^{-2u^2-2i\pi u} = e^{-\frac{\pi^2}{2}}\; e^{-2\left(u+\frac{i\pi}{2}\right)^2} \tag{12.16}$$

We define a shifted variable:

$$v = u + \frac{i\pi}{2} \tag{12.17}$$

Step 8: Getting a Gaussian integral
Substituting Eqs. (12.16) and (12.17) into Eq. (12.15), the integral becomes:

$$W_{mn} \propto \int_{-\infty}^{\infty} dv\; e^{-2v^2}\; \tilde{P}_{mn}(v,\xi',\pi) \tag{12.18}$$

This is a Gaussian integral of a polynomial, which yields Laguerre polynomials.
Step 9: Using Standard Result
Evaluating the integral in Eq. (12.18), we get:

$$W_{mn} \propto e^{-2(\xi'^2+\pi^2)}(\xi'+i\pi)^{n-m}L_m^{(n-m)}\big(4(\xi'^2+\pi^2)\big) \qquad n \geq m \tag{12.19}$$

Note that the associated Laguerre polynomial $L_m^{n-m}(x)$ is defined when $n \geq m$. Taking complex conjugate of Eq. (12.9), we get:

$$W^*_{mn} \propto e^{-2(\xi'^2+\pi^2)}(\xi'-i\pi)^{n-m}L_m^{(n-m)}\big(4(\xi'^2+\pi^2)\big) \tag{12.20}$$

Using Eq. (9.1), we know, $W^*_{mn} = W_{nm}$. Substituting this in Eq. (12.20), we get:

$$W_{nm} \propto e^{-2(\xi'^2+\pi^2)}(\xi'-i\pi)^{n-m}L_m^{(n-m)}\big(4(\xi'^2+\pi^2)\big) \tag{12.21}$$

Re-indexing $(m, n) \to (n, m)$ in Eq. (12.21), we get:

$$W_{mn} \propto e^{-2(\xi'^2+\pi^2)}(\xi' - i\pi)^{m-n} L_n^{(m-n)}\big(4(\xi'^2 + \pi^2)\big) \qquad n < m \tag{12.22}$$

Step 10: Final Result

Using Eqs. (12.19) and (12.22) and including normalization constants, we have:

$$W_{mn}(x,p) = \frac{2(-1)^m}{\pi\hbar}\sqrt{\frac{m!}{n!}}\, e^{-2(\xi'^2+\pi^2)}\big(2(\xi' + i\pi)\big)^{n-m} L_m^{(n-m)}\big(4(\xi'^2 + \pi^2)\big) \qquad n \geq m$$

and

$$W_{mn}(x,p) = W^*_{nm}(x,p) \qquad m > n$$

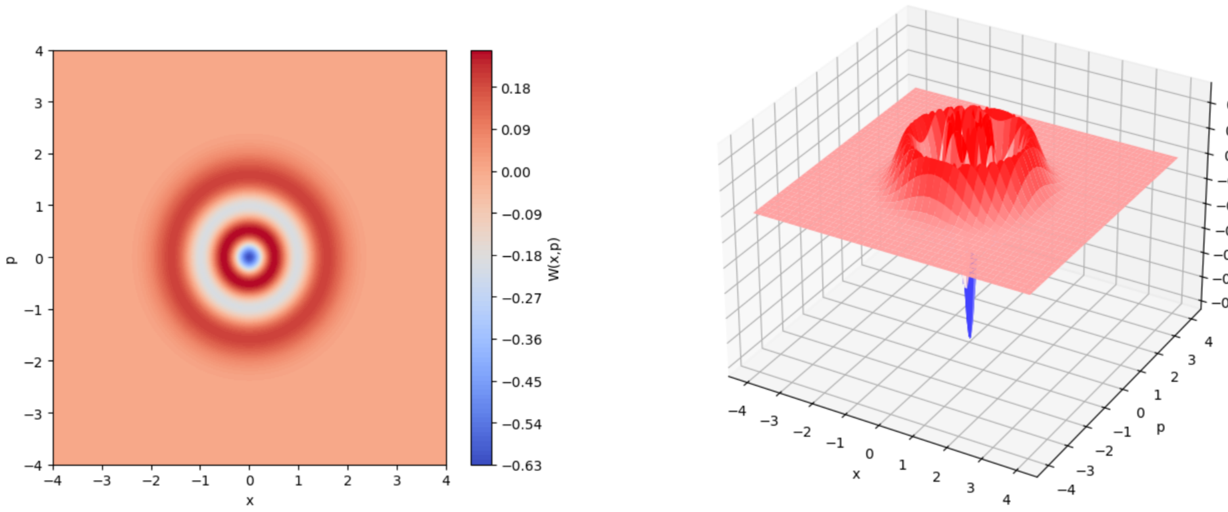
where

$$\xi' = \frac{x}{x_0}, \qquad \pi = \frac{p x_0}{\hbar}$$

13 WIGNER PLOTS AND INFERENCES

We will use the Equation for Wigner Functions derived in previous sections.
Note: Refer to the appendix for the link to the codes.

13.1 Fock State $|3\rangle$



(a) Wigner Plot (b) Wigner 3D Plot

Figure 1: Fock State $|3\rangle$ Plots

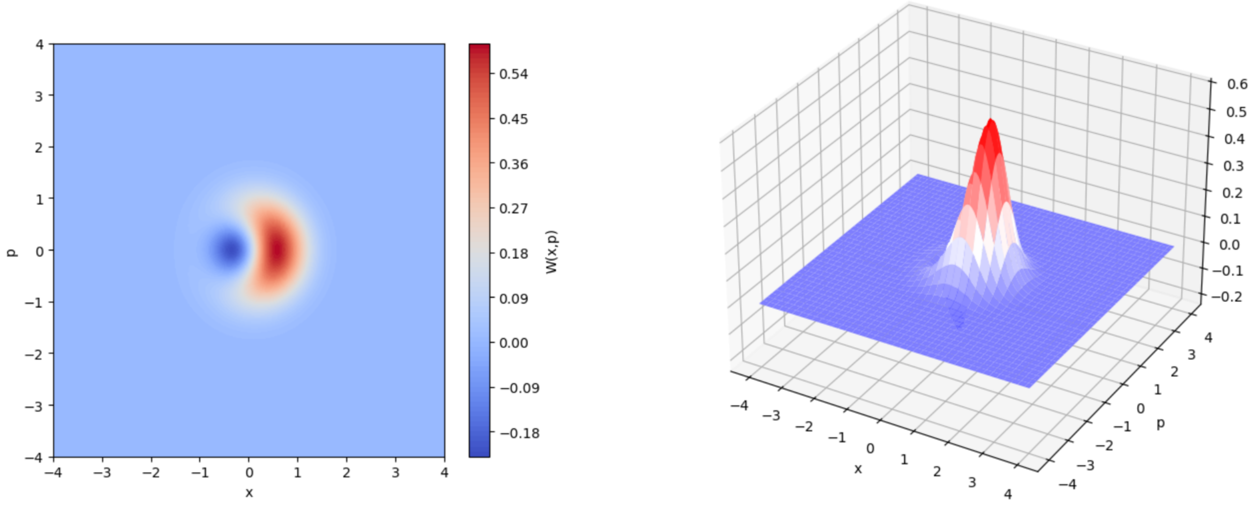
1. **Radial symmetry:** The Wigner distribution is rotationally symmetric in phase space, indicating that the state depends only on the radial coordinate

$$r^2 = \xi'^2 + \pi^2,$$

and possesses no preferred phase direction.

2. **Absence of definite phase:** Since number states are eigenstates of the Hamiltonian but not of phase, the circular symmetry reflects complete phase indeterminacy.
3. **Strong nonclassicality:** The central negative region and alternating positive/negative rings indicate negativity of the Wigner function, which has no classical probabilistic analogue.
4. **Quantum interference in phase space:** Oscillatory ring structures arise from Laguerre polynomial contributions, representing self-interference between quadrature amplitudes.
5. **Purity of the quantum state:** Sharp oscillatory structures and well-defined negativity indicate a pure quantum state with maximal coherence.

13.2 Superposition State $(|0\rangle + |1\rangle)\sqrt{2}$



(a) Wigner Plot

(b) Wigner 3D Plot

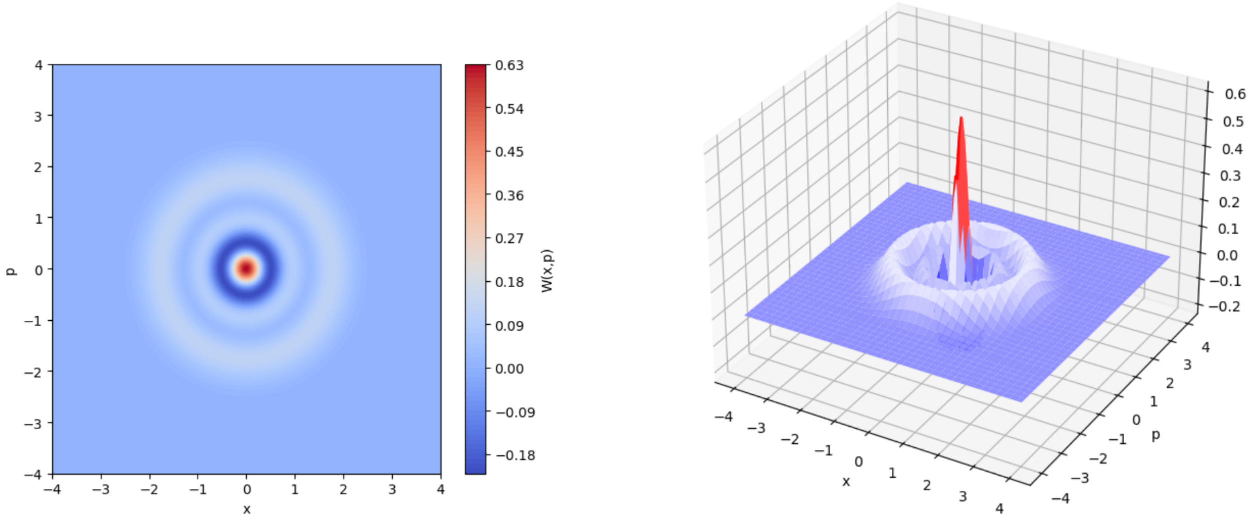
Figure 2: Superposition state: $(|0\rangle + |1\rangle)\sqrt{2}$

1. **Broken radial symmetry:** Unlike number states, the Wigner function is asymmetric, indicating the presence of a preferred phase-space direction.
2. **Presence of coherence:** The asymmetry originates from off-diagonal density matrix elements

   $$\rho_{01}, \rho_{10},$$

   which represent quantum coherence between vacuum and single-photon states.
3. **Interference fringes:** Positive and negative lobes correspond to interference between the constituent basis states.
4. **Phase sensitivity:** The orientation of lobes encodes relative phase information between superposed states.
5. **Manifestation of superposition principle:** The Wigner function is not simply the sum of two Gaussian-like distributions; additional interference terms appear explicitly.
6. **Quantum coherence visualization:** This plot directly visualizes coherence in phase space, making Wigner functions particularly useful for studying superposed states.

13.3 Mixed State $0.5|2\rangle\langle 2| + 0.5|4\rangle\langle 4|$



(a) Wigner Plot

(b) Wigner 3D Plot

Figure 3: Mixed State $0.5|2\rangle\langle 2| + 0.5|4\rangle\langle 4|$

1. **Convex combination structure:** The Wigner function is a weighted sum

$$W = 0.5\,W_{22} + 0.5\,W_{44},$$

reflecting statistical mixing rather than coherent superposition.

2. **Retention of radial symmetry:** Since both contributing states are number states, circular symmetry is preserved.
3. **Suppressed negativity:** Negative regions are weaker and smoother due to averaging over multiple states.
4. **Partial classicalization:** The reduced oscillatory contrast indicates a transition toward more classical behavior.
5. **Energy uncertainty:** The state is not an eigenstate of photon number, leading to broader radial spread in phase space.
6. **Signature of decoherence:** This state models decoherence effects, where populations are preserved but phase coherence is lost.

14 QUANTUM TOMOGRAPHY AND THE RADON TRANSFORM

14.1 Quadrature Measurements

Definition 14.1. Quadrature Operator at Phase θ

$$\hat{X}_\theta = \frac{1}{\sqrt{2}}(\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta})$$

This measures a linear combination of position and momentum at angle θ in phase space.

Examples:
- $\theta = 0$: $\hat{X}_0 = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger) \propto \hat{x}$ (position)
- $\theta = \pi/2$: $\hat{X}_{\pi/2} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^\dagger) \propto \hat{p}$ (momentum)

14.2 Connection to Radon Transform

Theorem 14.1. Radon Transform of Wigner

The probability distribution for measuring $\hat{X}_\theta$ is:

$$P_\theta(u) = \int dv\, W(u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta)$$

This is the *Radon transform* of the Wigner function at angle θ and position u.

Proof. The probability distribution for measuring $\hat{X}_\theta$ is

$$P_\theta(u) = \mathrm{Tr}\big[\rho\, \delta(u - \hat{X}_\theta)\big].$$

Using the Fourier representation of the delta function,

$$\delta(u - \hat{X}_\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ik(u-\hat{X}_\theta)},$$

we obtain

$$P_\theta(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{iku}\; \mathrm{Tr}\big[\rho\, e^{-ik\hat{X}_\theta}\big].$$

Using the Weyl correspondence,

$$\mathrm{Tr}(\rho\hat{A}) = \int dx\, dp\, W(x,p)\, A_W(x,p),$$

we need the Weyl symbol of $e^{-ik\hat{X}_\theta}$. Since $\hat{X}_\theta$ is linear in $\hat{x}$ and $\hat{p}$, its Weyl symbol is

$$(e^{-ik\hat{X}_\theta})_W = e^{-ik(x\cos\theta + p\sin\theta)}.$$

Therefore,

$$\mathrm{Tr}\big[\rho e^{-ik\hat{X}_\theta}\big] = \int dx\,dp\,W(x,p)\,e^{-ik(x\cos\theta+p\sin\theta)}.$$

Substituting back,

$$P_\theta(u) = \frac{1}{2\pi}\int dk\;e^{iku}\int dx\,dp\,W(x,p)\,e^{-ik(x\cos\theta+p\sin\theta)}.$$

Interchanging the order of integration (assuming sufficient convergence),

$$P_\theta(u) = \int dx\,dp\,W(x,p)\left[\frac{1}{2\pi}\int dk\;e^{ik(u-x\cos\theta-p\sin\theta)}\right].$$

Recognizing the delta function,

$$\frac{1}{2\pi}\int dk\;e^{ik(u-x\cos\theta-p\sin\theta)} = \delta(u-x\cos\theta-p\sin\theta),$$

we obtain

$$P_\theta(u) = \int dx\,dp\,W(x,p)\,\delta(u-x\cos\theta-p\sin\theta).$$

Now perform a change of variables:

$$\begin{cases} u = x\cos\theta + p\sin\theta, \\ v = -x\sin\theta + p\cos\theta, \end{cases}$$

which is a rotation with Jacobian equal to 1. Hence $dx\,dp = du\,dv$.
Using the delta function to fix u, we get

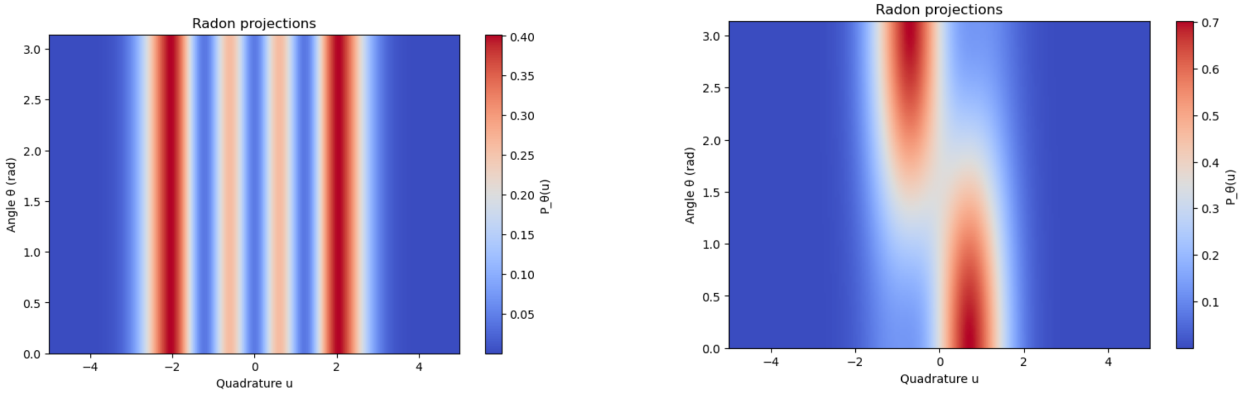
$$P_\theta(u) = \int_{-\infty}^{\infty} dv\;W(u\cos\theta - v\sin\theta,\; u\sin\theta + v\cos\theta).$$

This completes the proof. □

Physical Interpretation: By measuring quadratures at different angles θ and reconstructing the probabilities, we obtain projections of the Wigner function.

14.3 Radon Transform Plots and Inference

We will use plot Radon Transform for the following states.
Note: Refer to the appendix for the link to the codes.



(a) Radon Projection for $|3\rangle$ fock state

(b) Radon Projection for $(|0\rangle + |1\rangle)/\sqrt{2}$ state

Figure 4: Radon Projections

(a) Radon Projection for $|3\rangle$ Fock State

1. The plot shows multiple parallel vertical bands that are largely independent of the angle θ.
2. This indicates that the quadrature distribution is rotationally symmetric in phase space, as expected for Fock states.
3. Alternating high and low intensity fringes reflect the non-classical oscillatory structure of the state.
4. The symmetry and lack of angular dependence imply that the state has no preferred phase direction.
5. These oscillations are signatures of quantum interference and relate to negativity in the Wigner function.

(b) Radon Projection for $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$

1. The plot exhibits clear angular dependence, with features shifting as θ varies.
2. This indicates that the state is not rotationally symmetric.
3. The asymmetry arises from quantum superposition, introducing a preferred phase.
4. The plot reflects interference between the $|0\rangle$ and $|1\rangle$ components.
5. The distribution is more localized and phase-sensitive compared to the Fock state.

14.4 Inverse Radon Transform for Wigner Reconstruction

Theorem 14.2. Inverse Radon Formula

From measurements $\{P_\theta(u) : \theta \in [0, \pi)\}$ at all angles, the Wigner function can be reconstructed:

$$W(x,p) = \frac{1}{2\pi} \int_0^\pi d\theta \int_{-\infty}^{\infty} dk \, |k| \, e^{ik(x\cos\theta + p\sin\theta)} \, \tilde{P}_\theta(k)$$

where $\tilde{P}_\theta(k) = \int du P_\theta(u) e^{-iku}$ is the Fourier transform of the marginal probability.

Proof. Define the Fourier transform of the marginal distribution:

$$\tilde{P}_\theta(k) = \int_{-\infty}^{\infty} du \; P_\theta(u) \, e^{-iku}.$$

Substitute the Radon transform expression:

$$P_\theta(u) = \int_{-\infty}^{\infty} dv \; W(u\cos\theta - v\sin\theta, \; u\sin\theta + v\cos\theta).$$

Introduce the change of variables

$$\begin{cases} x = u\cos\theta - v\sin\theta, \\ p = u\sin\theta + v\cos\theta, \end{cases}$$

which has unit Jacobian, so $dx\,dp = du\,dv$.
Thus,

$$\tilde{P}_\theta(k) = \int dx\,dp\,W(x,p)\,e^{-ik(x\cos\theta+p\sin\theta)}.$$

Define the two-dimensional Fourier transform of $W(x,p)$:

$$\tilde{W}(k_x,k_p) = \int dx\,dp\,W(x,p)\,e^{-i(k_x x+k_p p)}.$$

Comparing, we see that

$$\tilde{P}_\theta(k) = \tilde{W}(k\cos\theta,\ k\sin\theta).$$

This is the central slice theorem: $\tilde{P}_\theta(k)$ gives the value of the Fourier transform of W along a radial line in Fourier space.
We now invert the Fourier transform:

$$W(x,p) = \frac{1}{2\pi}\int dk_x\,dk_p\,\tilde{W}(k_x,k_p)\,e^{i(k_x x+k_p p)}.$$

Changing to polar coordinates:

$$k_x = k\cos\theta, \quad k_p = k\sin\theta,$$

with Jacobian

$$dk_x\,dk_p = |k|\,dk\,d\theta.$$

Substituting and using $\tilde{W}(k_x,k_p) = \tilde{P}_\theta(k)$, we obtain

$$W(x,p) = \frac{1}{2\pi}\int_0^\pi d\theta\int_{-\infty}^{\infty} dk\,|k|\,e^{ik(x\cos\theta+p\sin\theta)}\tilde{P}_\theta(k).$$

This completes the proof. ☐

14.5 Homodyne Tomography - Qualitative Explanation

The Homodyne Detection Setup:

1. **Unknown quantum field**: State with unknown density operator $\hat{\rho}$
2. **Local oscillator**: Strong classical reference field at fixed phase
3. **Beam splitter**: We mix unknown and reference
4. **Photodiode**: We measure intensity difference which is equivalent to measuring quadrature $\hat{X}_\theta$
5. **Phase scanning**: We vary phase of local oscillator to measure different angles θ
6. **Reconstruction**: We apply inverse Radon transform to obtain Wigner function

Experimental Procedure:

For each phase θ:

1. We Measure quadrature $\hat{X}_\theta$ many times
2. We record histogram of results which gives a measure of $P_\theta(u)$

After scanning all phases:

1. We apply inverse Radon transform
2. We obtain complete Wigner function $W(x,p)$

This is the *gold standard* for quantum state characterization!

15 TIME EVOLUTION OF THE WIGNER FUNCTION

15.1 Quantum Dynamics of the Density Operator

The time evolution of a quantum state described by the density operator $\hat{\rho}(t)$ is governed by the von Neumann equation:

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{1}{i\hbar}[\hat{H},\hat{\rho}] = \frac{1}{i\hbar}(\hat{H}\hat{\rho} - \hat{\rho}\hat{H}) \tag{15.1.1}$$

where $\hat{H}$ is the Hamiltonian operator.

Our goal is to derive the corresponding evolution equation for the Wigner function.

15.2 Definition of the time dependent Wigner Function

Using Eq. (6.3), the time dependent Wigner function is defined as:

$$W(x,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\xi\, e^{-ip\xi/\hbar}\,\rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}, t\right) \tag{15.2.1}$$

15.3 Time Derivative of the Wigner Function

Differentiating Eq. (15.2.1) with respect to time:

$$\frac{\partial W}{\partial t} = \frac{1}{2\pi\hbar}\int d\xi\, e^{-ip\xi/\hbar}\frac{\partial}{\partial t}\rho\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}, t\right) \tag{15.3.1}$$

Substituting Eq. (15.1.1) in Eq. (15.3.1), we have:

$$\frac{\partial W}{\partial t} = \frac{1}{2\pi\hbar}\int d\xi\, e^{-ip\xi/\hbar}\frac{1}{i\hbar}\Big(\langle x+\tfrac{\xi}{2}|\hat{H}\hat{\rho}|x-\tfrac{\xi}{2}\rangle - \langle x+\tfrac{\xi}{2}|\hat{\rho}\hat{H}|x-\tfrac{\xi}{2}\rangle\Big) \tag{15.3.2}$$

15.4 Hamiltonian Decomposition

We consider a Hamiltonian of the form:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

We evaluate the kinetic and potential contributions to Eq. (15.3.2) separately.

Kinetic Contribution

$$\left(\frac{\partial W}{\partial t}\right)_{\text{kin}} = \frac{1}{2\pi\hbar}\int d\xi\, e^{-ip\xi/\hbar}\frac{1}{i\hbar}\frac{1}{2m}\Big(\langle x+\tfrac{\xi}{2}|\hat{p}^2\hat{\rho}|x-\tfrac{\xi}{2}\rangle - \langle x+\tfrac{\xi}{2}|\hat{\rho}\hat{p}^2|x-\tfrac{\xi}{2}\rangle\Big) \tag{15.4.1}$$

We define:

$$x = \frac{q_1 + q_2}{2}, \qquad \xi = q_1 - q_2$$

Using

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \quad \Rightarrow \quad \hat{p}^2 = -\hbar^2\frac{\partial^2}{\partial x^2}$$

in Eq. (15.4.1), we get:

$$\langle q_1|\hat{p}^2\hat{\rho}|q_2\rangle = -\hbar^2\frac{\partial^2}{\partial q_1^2}\rho(q_1, q_2)$$
$$\langle q_1|\hat{\rho}\hat{p}^2|q_2\rangle = -\hbar^2\frac{\partial^2}{\partial q_2^2}\rho(q_1, q_2)$$

Thus,

$$\langle q_1|\hat{p}^2\hat{\rho}|q_2\rangle - \langle q_1|\hat{\rho}\hat{p}^2|q_2\rangle = -\hbar^2\left(\frac{\partial^2}{\partial q_1^2} - \frac{\partial^2}{\partial q_2^2}\right)\rho(x_1, x_2)$$

Then:

$$\frac{\partial}{\partial q_1} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{\partial}{\partial \xi}$$
$$\frac{\partial}{\partial q_2} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{\partial}{\partial \xi}$$

Hence:

$$\frac{\partial^2}{\partial q_1^2} - \frac{\partial^2}{\partial q_2^2} = 2\frac{\partial}{\partial x}\frac{\partial}{\partial \xi}$$

So the kinetic term becomes:

$$\left(\frac{\partial W}{\partial t}\right)_{\text{kin}} = -\frac{1}{2\pi i m}\int d\xi\, e^{-ip\xi/\hbar}\frac{\partial}{\partial x}\frac{\partial}{\partial \xi}\rho$$

Using Integration by parts:

$$\int d\xi\, e^{-ip\xi/\hbar}\frac{\partial \rho}{\partial \xi} = \frac{ip}{\hbar}\int d\xi\, e^{-ip\xi/\hbar}\rho$$

Thus,

$$\left(\frac{\partial W}{\partial t}\right)_{\text{kin}} = -\frac{p}{m}\frac{\partial W}{\partial x}$$

Potential Term

$$\left(\frac{\partial W}{\partial t}\right)_{\text{pot}} = \frac{1}{2\pi\hbar}\int d\xi\, e^{-ip\xi/\hbar}\frac{1}{i\hbar}\left[V\left(x + \frac{\xi}{2}\right) - V\left(x - \frac{\xi}{2}\right)\right]\rho$$

Expanding:

$$V\left(x \pm \frac{\xi}{2}\right) = \sum_{n=0}^{\infty} \frac{(\pm\xi/2)^n}{n!} \frac{d^n V}{dx^n}$$

Subtracting:

$$V\left(x + \frac{\xi}{2}\right) - V\left(x - \frac{\xi}{2}\right) = 2\sum_{n=0}^{\infty} \frac{(\xi/2)^{2n+1}}{(2n+1)!} \frac{d^{2n+1} V}{dx^{2n+1}}$$

Substituting:

$$\left(\frac{\partial W}{\partial t}\right)_{\text{pot}} = \sum_{n=0}^{\infty} \frac{2}{(2n+1)!} \left(\frac{1}{2}\right)^{2n+1} \frac{d^{2n+1} V}{dx^{2n+1}} \int d\xi \, e^{-ip\xi/\hbar} \xi^{2n+1} \rho$$

Using:

$$\xi^{2n+1} e^{-ip\xi/\hbar} = (i\hbar)^{2n+1} \frac{\partial^{2n+1}}{\partial p^{2n+1}} e^{-ip\xi/\hbar}$$

we obtain:

$$\int d\xi \, e^{-ip\xi/\hbar} \xi^{2n+1} \rho = (i\hbar)^{2n+1} \frac{\partial^{2n+1} W}{\partial p^{2n+1}}$$

Thus:

$$\left(\frac{\partial W}{\partial t}\right)_{\text{pot}} = \sum_{n=0}^{\infty} \frac{(-1)^n (\hbar/2)^{2n}}{(2n+1)!} \frac{d^{2n+1} V}{dx^{2n+1}} \frac{\partial^{2n+1} W}{\partial p^{2n+1}}$$

15.5 Final Wigner Equation

Combining both contributions:

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial x} + \sum_{n=0}^{\infty} \frac{(-1)^n (\hbar/2)^{2n}}{(2n+1)!} \frac{d^{2n+1} V}{dx^{2n+1}} \frac{\partial^{2n+1} W}{\partial p^{2n+1}}$$

This is the exact evolution equation of the Wigner function, also known as the quantum Liouville equation.

15.6 Classical Limit

In the limit $\hbar \to 0$, higher-order terms vanish and we obtain:

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial x} + \frac{dV}{dx}\frac{\partial W}{\partial p}$$

which is precisely the classical Liouville equation.

15.7 Physical Interpretation

The Wigner function evolves as a quasi-probability distribution in phase space:

1. The first term describes classical transport in phase space
2. Higher-order terms encode quantum corrections
3. These corrections generate interference and negativity

15.8 Harmonic Oscillator Wigner Equation

For the harmonic oscillator:

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

Its derivatives are:

$$\begin{aligned} V'(x) &= m\omega^2 x \\ V''(x) &= m\omega^2 \\ V^{(n)}(x) &= 0 \quad \text{for } n \geq 3 \end{aligned}$$

Since the Wigner equation contains only odd derivatives $V^{(2n+1)}(x)$, all terms vanish except $n = 0$. Thus the potential term reduces to:

$$V'(x)\frac{\partial W}{\partial p} = m\omega^2 x\frac{\partial W}{\partial p}$$

Combining with the kinetic term:

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial x} + m\omega^2 x\frac{\partial W}{\partial p} \tag{15.9.1}$$

15.9 Wigner Evolution for Harmonic Oscillator

Eq. (15.9.1) could be written as:

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x}\frac{p}{m} - \frac{\partial W}{\partial p}m\omega^2 x = 0 \qquad \Rightarrow \qquad \frac{dW}{dt} = 0 \tag{15.10.1}$$

where using standard restults, we get:

$$\frac{dx}{dt} = \frac{p}{m} \qquad \Rightarrow \qquad x = x_0\cos(\omega t) + \frac{p_0}{m\omega}\sin(\omega t) \tag{15.10.2}$$

$$\frac{dp}{dt} = -m\omega^2 x \qquad \Rightarrow \qquad p = p_0\cos(\omega t) - m\omega x_0\sin(\omega t) \tag{15.10.3}$$

Solution of Eq. (15.10.1) has the form:

$$W(x, p, t) = W(x_0, p_0, 0)$$

where x and p are given by Eqs. (15.10.2) and (15.10.3) respectively.

15.10 Physical Insights

1. The Wigner function preserves its shape during evolution; coherent-state Gaussians remain minimum-uncertainty packets.
2. Constant-energy trajectories are ellipses in phase space:

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \text{constant}.$$

3. Quantum interference and negativity of the Wigner function may persist, but they are transported rigidly along the classical flow.

APPENDIX

1. Normalization

The trace of the density operator $\hat{\rho}$ is defined as:

$$\mathrm{Tr}(\hat{\rho}) = \int_{-\infty}^{\infty} \langle x|\hat{\rho}|x\rangle \, dx$$

In the position representation:

$$\rho(x, x) = \langle x|\hat{\rho}|x\rangle$$

Thus,

$$\mathrm{Tr}(\hat{\rho}) = \int_{-\infty}^{\infty} \rho(x, x) \, dx$$

Since $\rho(x, x)$ represents the probability density, normalization requires:

$$\int_{-\infty}^{\infty} \rho(x, x) \, dx = 1$$

Therefore,

$$\mathrm{Tr}(\hat{\rho}) = 1$$

2. Expectation Value

(a) Pure State

For a pure state $|\psi\rangle$, the expectation value of an operator $\hat{A}$ is:

$$\langle \hat{A}\rangle = \langle \psi|\hat{A}|\psi\rangle$$

The density operator is:

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

Now computing the trace:

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \sum_n \langle n|\hat{\rho}\hat{A}|n\rangle$$

Inserting $\hat{\rho}$, we get:

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \sum_n \langle n|\psi\rangle\langle\psi|\hat{A}|n\rangle$$

Rearranging, we get:

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \sum_n \langle\psi|\hat{A}|n\rangle\langle n|\psi\rangle = \langle\psi|\hat{A}|\psi\rangle$$

Thus,

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \langle\hat{A}\rangle$$

(b) Mixed State

For a mixed state:

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Then, we have:

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \sum_i p_i \mathrm{Tr}(|\psi_i\rangle\langle\psi_i|\hat{A})$$

Using the pure state result:

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \sum_i p_i \langle\psi_i|\hat{A}|\psi_i\rangle$$

Hence,

$$\langle\hat{A}\rangle = \mathrm{Tr}(\hat{\rho}\hat{A})$$

3. Link to codes for the plots used in this paper

https://github.com/KishalTandel/Research-Work/tree/main/Quasiprobabilities

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